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Elastic deformation of the rotating functionally graded annular disk with rigid casing

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Abstract An accurate solution for a rotating functionally graded annular disk is presented. Material properties of the present annular disk are assumed to be graded in the radial direction according to a simple exponential-law distribution. The inner surface of the disk is pure metal whereas the outer surface of the disk is pure ceramic. The boundary condition of rigid casing is considered herein, that is the vanishing of the radial displacement at the outer surface. The boundary condition at the inner surface of the disk is taken to be vanishing either radial displacement or radial stress. Analytical solutions for the elastic deformation of the rotating functionally graded annular disks subjected to these boundary conditions are obtained. Numerical results for radial displacement, circumferential and radial stresses are presented. Comparisons between the different rotating homogeneous and functionally graded annular disks are made at the same angular velocity. The results show that distributions of stresses and displacement through the radial direction of the rotating annular disk vary with different parameters.

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Introduction

The analysis of rotating annular disks is an important subject for many applications in mechanical engineering. Among these high-speed gears, turbine rotors, flywheels, disc brakes, disk drives, circular saw blades and shrink fits can be mentioned. The research on them is always an important topic and their benefits have been included in many references [1, 2]. The problem of determination of stresses and displacement in rotating annular disks subjected to different boundary conditions is important for an efficient design and material usage in many industrial applications. Most of the research works are concentrated on the analytical solutions of rotating disks with simple cross-section geometries of constant thickness and specifically variable thickness. The material density of these rotating disks is taken to be either constant or specifically variable. The analytical elasticity solutions of such rotating disks can be mainly found in the literature.

Gamer [3] has studied the analytical solution of elastic-perfectly plastic rotating disks of constant thickness and density by using Tresca's yield condition. Gamer [4, 5] has also studied the analytical solutions of such disks with a linear strain-hardening material behaviour using the same yield condition. Güven [6] has extended this work to rotating disks of variable thickness and density and obtained their analytical solution. You and Zhang [7] have obtained the approximate analytical solution for a rotating solid disk of uniform thickness. You et al. [8, 9] have numerically studied rotating solid disks of uniform thickness and constant density and annular disks of variable thickness and variable density. Eraslan [10, 11], and Eraslan and Orcan [12] have analytically studied rotating disks of exponentially varying thickness and of linearly strain hardening material. In a recent paper, Zenkour and

Allam [13] have developed analytical solutions for the analysis of deformation and stresses in elastic rotating viscoelastic solid and annular disks with arbitrary cross-sections of continuously variable thickness.

In recent years, functionally graded materials (FGMs) have gained considerable attention in many engineering applications. FGMs are considered as a potential structural material for future high-speed spacecraft and power generation industries. FGMs are new materials, microscopically inhomogeneous, in which the mechanical properties vary smoothly and continuously from one surface to the other. In an FGM, the composition and structure gradually change over volume, resulting in corresponding changes in the properties of the material. By applying the many possibilities inherent in the FGM concept, it is anticipated that materials will be improved and new functions for them established.

In the simplest FGMs, two different material ingredients change gradually from one to the other. Discontinuous changes such as a stepwise gradation of the material ingredients can also be considered an FGM. The most familiar FGM is compositionally graded from a refractory ceramic to a metal. The ceramic in an FGM offers thermal barrier effects and protects the metal from corrosion as well as oxidation and the FGM is toughened and strengthened by the metallic composition. A mixture of ceramic and metal with a continuously varying volume fraction can be easily manufactured. This eliminates interface problems of composite materials and thus the stress distributions are smooth.

The analysis of rotating such functionally graded material (FGM) disks has been reported rarely in the literature. Horgan and Chan [14, 15] have investigated the pressured FGM hollow cylinder and disk problems and the stress response of FGM isotropic linear elastic rotating disk. Based on the assumption of a generalized plane strain, Ha et al. [16] have calculated the stress and strength ratio distributions of the rotating composite flywheel rotor of varying material properties in the radial direction. In addition, Zenkour [17] has presented accurate elastic solutions for exponentially graded rotating annular disks with various boundary conditions. In fact, the above articles discussed the general inhomogeneous materials and they were unsuitable for functionally graded two phases metal–ceramic material.

The objective of the present study is: to develop an analytical solution for rotating functionally graded metal–ceramic annular disk; to find a closed form solution for the rotating disk taking into account the rigid casing condition; and to investigate the effect of material properties on the distributions of displacement and stresses through the radial direction of the annular disk. These objectives are all in order to give benchmark results for future comparison of such FG metal–ceramic annular disk.

Theoretical formulation

Let us consider a functionally graded annular disk of inner radius a and outer radius b. The material properties of the annular disk are assumed to be functions of the volume fraction of the constituent materials. The functional relationships between Young's modulus E and r and between the density ρ and r, of the disk are assumed to be

$$E = E_m e^{k(r-a)/b}, \quad \rho = \rho_m e^{j(r-a)/b}, \tag{1}$$

where E_m and ρ_m are Young's modulus and density of the metal material, respectively, r is the radial coordinate, and k and j are given by

$$k = \frac{b}{b-a}\ln\left(\frac{E_c}{E_m}\right), \quad j = \frac{b}{b-a}\ln\left(\frac{\rho_c}{\rho_m}\right), \tag{2}$$

in which E_c and ρ_c are Young's modulus and density of the ceramic material, respectively. Equations (1) and (2) reflect a simple rule of mixtures used to obtain the effective properties of the metal–ceramic annular disk. The rule of mixtures applies only to the radial direction. Note that the volume fraction of the metal is high near the inner surface of the annular disk, and that of ceramic high near the outer surface. In addition, these equations indicate that the inner surface of the annular disk ($r = a$) is pure metal whereas the outer surface of the annular disk ($r = b$) is pure ceramic.

The effect of material properties variation of the rotating disk can be taken into account in its equation of equilibrium, which can be written as

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta + \rho\omega^2 r^2 = 0, \tag{3}$$

where σ_r and σ_θ are the radial and circumferential stresses, respectively, and ω is the constant angular velocity.

The stress components are depending on the modulus of elasticity of the material of the rotating disk. They can be written as

$$\begin{aligned}\sigma_r &= \frac{E}{1-\nu^2}\left(\frac{du_r}{dr} + \nu\frac{u_r}{r}\right),\\ \sigma_\theta &= \frac{E}{1-\nu^2}\left(\frac{u_r}{r} + \nu\frac{du_r}{dr}\right),\end{aligned} \tag{4}$$

where ν is Poisson's ratio.

Elastic solution

The substitution of Eq. (4) into Eq. (3) produces the following differential equation:

$$r^2 \frac{d^2 u_r}{dr^2} + r\left(1 + \frac{r}{E}\frac{dE}{dr}\right)\frac{du_r}{dr} - \left(1 - v\frac{r}{E}\frac{dE}{dr}\right)u_r + \frac{(1-v^2)\rho\omega^2 r^3}{E} = 0. \quad (5)$$

The above equation with the aid of Eq. (1) tends to the following differential equation for the radial displacement u_r:

$$r^2 \frac{d^2 u_r}{dr^2} + r\left(1 + k\frac{r}{b}\right)\frac{du_r}{dr} - \left(1 - kv\frac{r}{b}\right)u_r + \frac{(1-v^2)e^{(r-a)(j-k)/b}\rho_m\omega^2 r^3}{E_m} = 0. \quad (6)$$

Introducing the following dimensionless forms:

$$\bar{r} = r/b,$$

$$\Omega = \omega b\sqrt{\rho_m(1-v^2)},$$

$$U(\bar{r}) = \frac{E_m}{b\Omega^2}u_r(r),$$

$$\{\Sigma_r, \Sigma_\theta\} = \frac{1-v^2}{\Omega^2}\{\sigma_r, \sigma_\theta\}. \quad (7)$$

Then, Eq. (6) may be written in the following simple form

$$\bar{r}^2 \frac{d^2 U}{d\bar{r}^2} + \bar{r}(1 + k\bar{r})\frac{dU}{d\bar{r}} - (1 - kv\bar{r})U + \bar{r}^3 e^{(\bar{r}-\bar{a})(j-k)} = 0. \quad (8)$$

The general solutions of the above equation can be derived in a similar manner as that given in Zenkour [17]. The elastic solution for the metal–ceramic annular disk is completed by the application of the boundary edge conditions. Here, we will investigate the elastic solution for the rotating metal–ceramic annular disk with rigid casing. In this case, the radial displacement at the outer surface of the disk ($r = b$ or $\bar{r} = 1$) should be vanished, $u_r(b) = 0$ or $U(1) = 0$.

Two types of a rotating metal–ceramic annular disk are presented depending on the boundary conditions at the inner surface of the disk.

Clamped annular disk

The inner surface of the annular disk ($r = a$ or $\bar{r} = \bar{a} = a/b$) is clamped, allowing no motion in radial direction, i.e. $u_r(a) = 0$ or $U(\bar{a}) = 0$.

Free annular disk

In this case, the inner surface $r = a$ ($\bar{r} = \bar{a}$) is free of constraint, $\sigma_r(a) = 0$ or $\Sigma_r(\bar{a}) = 0$.

Numerical examples and discussion

The functionally graded materials can be made by mixing two distinct materials such as a metal and a ceramic. The functionally graded disk is taken to be made of aluminum and zirconia with material properties [18]

$$E_m = 70\,\text{GPa}, \quad \rho_m = 2,707\,\text{kg/m}^3,$$
$$E_c = 151\,\text{GPa}, \quad \rho_c = 3,000\,\text{kg/m}^3.$$

For simplicity, Poisson's ratio is chosen as 0.3 for both aluminum and zirconia. Figure 1 shows, respectively, the dimensionless modulus of elasticity $E(r)/E_m$ and the dimensionless density $\rho(r)/\rho_m$ through the radial direction of the annular disk. Note that the inner surface of the functionally graded disk is pure metal (aluminum) while that of the outer surface is pure ceramic (zirconia).

We observe that Young's moduli and densities ratios of the aluminum–zirconia functionally graded material are

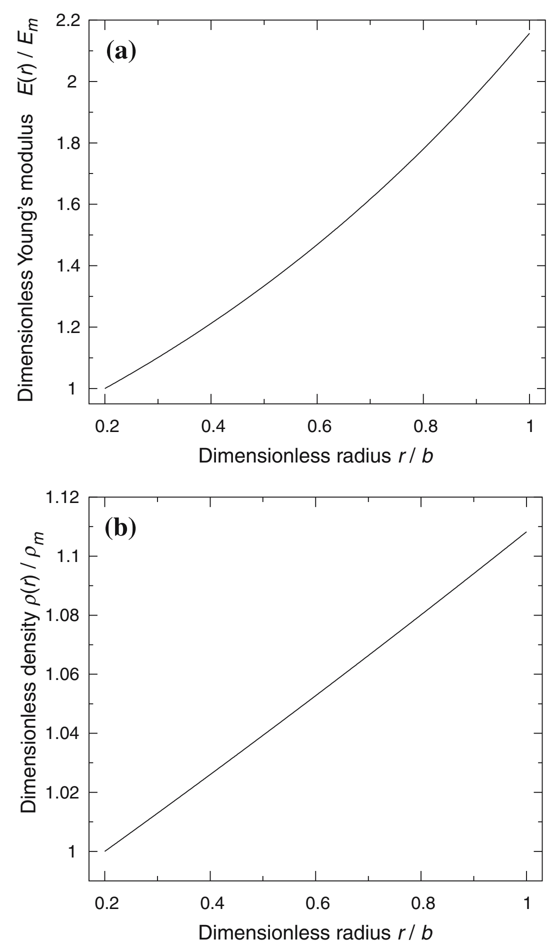

Fig. 1 Variation of material properties of aluminum–zirconia annular disk along the radial direction (**a**) Dimensionless Young's modulus and (**b**) Dimensionless density

around $E_c/E_m = 2.157$ and $\rho_c/\rho_m = 1.108$, respectively. Following these material properties ratios of the aluminum–zirconia functionally graded material, one can similarly achieve other metal–ceramic functionally graded materials having the following ratios:

$$\text{material 1}: E_c/E_m = 2.2, \rho_c/\rho_m = 1.11.$$
$$\text{material 2}: E_c/E_m = 1.8, \rho_c/\rho_m = 1.09.$$
$$\text{material 3}: E_c/E_m = 1.4, \rho_c/\rho_m = 1.07.$$

In addition, the material properties for a homogeneous isotropic disk ($k = 0$ and $j = 0$) are taken into account.

The numerical applications are carried out for the radial displacement and stresses that being reported herein are in the following dimensionless forms:

$$u = \frac{100E_m}{b\Omega^2}u_r(r) = 100U(\bar{r}),$$
$$\{\sigma_1, \sigma_2\} = \frac{10(1-\nu^2)}{\Omega^2}\{\sigma_r, \sigma_\theta\} = 10\{\Sigma_r, \Sigma_\theta\}.$$

The results of the present investigation are reported in Tables 1–6 and plotted in Figs. 2–5. The dimensionless radial parameter is given in Tables 1–6 according to the following formula:

$$\bar{r} = 1 + \bar{a} - e^{\beta \ln \bar{a}},$$

where β is a new factor used to make $\bar{r}$ ranging from the inner to the outer surfaces of the annular disk. For example, $\beta = 0$ tends to $\bar{r} = \bar{a}$ (the inner surface) while $\beta = 1$ tends to $\bar{r} = 1$ (the outer surface).

Table 1 Dimensionless radial displacements u for rotating clamped annular disks made of aluminum–zirconia FGM

b/a	β										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
4.5	0.0	1.92538	2.67193	2.84346	2.68598	2.34096	1.89738	1.41213	0.92107	0.44614	0.0
5.0	0.0	2.00946	2.75852	2.91150	2.72854	2.35938	1.89746	1.40145	0.90737	0.43639	0.0
5.5	0.0	2.07633	2.82487	2.96017	2.75436	2.36457	1.88806	1.38477	0.89052	0.42551	0.0
6.0	0.0	2.13046	2.87681	2.99535	2.76883	2.36118	1.87282	1.36471	0.87214	0.41425	0.0
6.5	0.0	2.17497	2.91824	3.02089	2.77542	2.35200	1.85402	1.34286	0.85321	0.40302	0.0
7.0	0.0	2.21214	2.95185	3.03940	2.77645	2.33906	1.83309	1.32019	0.83427	0.39205	0.0
7.5	0.0	2.24362	2.97954	3.05266	2.77349	2.32360	1.81096	1.29732	0.81565	0.38146	0.0
8.0	0.0	2.27064	3.00265	3.06195	2.76763	2.30647	1.78826	1.27462	0.79754	0.37131	0.0
8.5	0.0	2.29411	3.02218	3.06819	2.75968	2.28828	1.76539	1.25233	0.78005	0.36162	0.0
9.0	0.0	2.31473	3.03884	3.07206	2.75018	2.26946	1.74262	1.23059	0.76323	0.35240	0.0
9.5	0.0	2.33304	3.05320	3.07405	2.73955	2.25030	1.72013	1.20947	0.74708	0.34363	0.0
10.0	0.0	2.34947	3.06567	3.07456	2.72811	2.23101	1.69804	1.18903	0.73161	0.33530	0.0

Table 2 Dimensionless radial displacements u for rotating free annular disks made of aluminum–zirconia FGM

b/a	β										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
4.5	4.74444	4.36386	4.17642	3.84290	3.37098	2.81317	2.21752	1.61930	1.04194	0.49963	0.0
5.0	4.47669	4.15298	4.04555	3.75374	3.30005	2.75041	2.16095	1.57110	1.00590	0.47982	0.0
5.5	4.21991	3.96319	3.93097	3.67472	3.23507	2.69135	2.10709	1.52515	0.97172	0.46117	0.0
6.0	3.98008	3.79632	3.83284	3.60607	3.17662	2.63677	2.05671	1.48208	0.93979	0.44386	0.0
6.5	3.75897	3.65122	3.74960	3.54676	3.12425	2.58659	2.00985	1.44196	0.91013	0.42786	0.0
7.0	3.55639	3.52570	3.67919	3.49547	3.07720	2.54039	1.96632	1.40463	0.88261	0.41308	0.0
7.5	3.37124	3.41730	3.61963	3.45090	3.03471	2.49774	1.92580	1.36987	0.85707	0.39943	0.0
8.0	3.20208	3.32365	3.56911	3.41192	2.99609	2.45819	1.88797	1.33743	0.83331	0.38679	0.0
8.5	3.04738	3.24265	3.52611	3.37756	2.96074	2.42134	1.85257	1.30709	0.81115	0.37505	0.0
9.0	2.90568	3.17244	3.48938	3.34705	2.92817	2.38686	1.81932	1.27864	0.79045	0.36413	0.0
9.5	2.77561	3.11146	3.45787	3.31973	2.89798	2.35447	1.78800	1.25189	0.77105	0.35394	0.0
10.0	2.65596	3.05837	3.43071	3.29509	2.86982	2.32392	1.75842	1.22667	0.75284	0.34441	0.0

Table 3 Dimensionless radial stresses σ_1 for rotating clamped annular disks made of aluminum–zirconia FGM

b/a	β										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
4.5	2.02480	1.22249	0.68422	0.17499	−0.32449	−0.80780	−1.26657	−1.69466	−2.08854	−2.44684	−2.76977
5.0	2.03838	1.20381	0.66573	0.14914	−0.35985	−0.85087	−1.31373	−1.74168	−2.13135	−2.48189	−2.79417
5.5	2.04762	1.18632	0.64891	0.12513	−0.39256	−0.89014	−1.35584	−1.78263	−2.16744	−2.51002	−2.81198
6.0	2.05402	1.17038	0.63351	0.10258	−0.42318	−0.92637	−1.39400	−1.81894	−2.19854	−2.53323	−2.82534
6.5	2.05853	1.15606	0.61930	0.08123	−0.45205	−0.96010	−1.42897	−1.85157	−2.22581	−2.55281	−2.83560
7.0	2.06171	1.14326	0.60607	0.06088	−0.47943	−0.99170	−1.46126	−1.88120	−2.25005	−2.56962	−2.84364
7.5	2.06397	1.13183	0.59363	0.04138	−0.50551	−1.02146	−1.49126	−1.90834	−2.27182	−2.58427	−2.85005
8.0	2.06557	1.12160	0.58186	0.02262	−0.53042	−1.04959	−1.51930	−1.93335	−2.29156	−2.59718	−2.85522
8.5	2.06668	1.11242	0.57064	0.00453	−0.55429	−1.07627	−1.54559	−1.95653	−2.30958	−2.60869	−2.85946
9.0	2.06744	1.10414	0.55990	−0.01295	−0.57719	−1.10163	−1.57034	−1.97812	−2.32615	−2.61904	−2.86296
9.5	2.06793	1.09664	0.54955	−0.02988	−0.59921	−1.12580	−1.59372	−1.99831	−2.34145	−2.62841	−2.86589
10.0	2.06822	1.08981	0.53956	−0.46300	−0.62041	−1.14887	−1.61585	−2.01725	−2.35564	−2.63694	−2.86836

Table 4 Dimensionless radial stresses σ_1 for rotating free annular disks made of aluminum–zirconia FGM

b/a	β										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
4.5	0.0	0.27570	0.04331	−0.33196	−0.75923	−1.19865	−1.62857	−2.03663	−2.41604	−2.76356	−3.07826
5.0	0.0	0.36278	0.11712	−0.27878	−0.72479	−1.17833	−1.61689	−2.02818	−2.40592	−2.74764	−3.05324
5.5	0.0	0.43856	0.17655	−0.23923	−0.70206	−1.16753	−1.61268	−2.02550	−2.40038	−2.73567	−3.03215
6.0	0.0	0.50417	0.22421	−0.21033	−0.68822	−1.16379	−1.61392	−2.02705	−2.39831	−2.72692	−3.01448
6.5	0.0	0.56085	0.26237	−0.18970	−0.68109	−1.16524	−1.61910	−2.03164	−2.39882	−2.72070	−2.99967
7.0	0.0	0.60982	0.29285	−0.17551	−0.67900	−1.17048	−1.62707	−2.03838	−2.40119	−2.71641	−2.98720
7.5	0.0	0.65218	0.31714	−0.16634	−0.68073	−1.17849	−1.63702	−2.04662	−2.40491	−2.71362	−2.97664
8.0	0.0	0.68888	0.33641	−0.16109	−0.68533	−1.18849	−1.64834	−2.05588	−2.40959	−2.71198	−2.96764
8.5	0.0	0.72074	0.35160	−0.15893	−0.69210	−1.19992	−1.66057	−2.06580	−2.41492	−2.71122	−2.95992
9.0	0.0	0.74848	0.36346	−0.15920	−0.70050	−1.21236	−1.67339	−2.07614	−2.42070	−2.71113	−2.95326
9.5	0.0	0.77267	0.37258	−0.16140	−0.71013	−1.22548	−1.68656	−2.08669	−2.42677	−2.71156	−2.94747
10.0	0.0	0.79383	0.37944	−0.16513	−0.72067	−1.23905	−1.69991	−2.09733	−2.43301	−2.71239	−2.94241

Table 5 Dimensionless circumferential stresses σ_2 for rotating clamped annular disks made of aluminum–zirconia FGM

b/a	β										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
4.5	0.60744	0.92259	0.85739	0.68541	0.46935	0.23608	0.00051	−0.22834	−0.44501	−0.64638	−0.83093
5.0	0.61151	0.96615	0.88787	0.70141	0.47278	0.22988	−0.01196	−0.24373	−0.46022	−0.65873	−0.83825
5.5	0.61429	1.00255	0.91110	0.71157	0.47208	0.22125	−0.02534	−0.25875	−0.47414	−0.66926	−0.84359
6.0	0.61621	1.03313	0.92884	0.71754	0.46858	0.21113	−0.03906	−0.27324	−0.48694	−0.67840	−0.84760
6.5	0.61756	1.05896	0.94242	0.72046	0.46316	0.20013	−0.05280	−0.28711	−0.49875	−0.68646	−0.85068
7.0	0.61851	1.08090	0.95278	0.72113	0.45643	0.18865	−0.06635	−0.30034	−0.50970	−0.69365	−0.85309
7.5	0.61919	1.09962	0.96063	0.72014	0.44881	0.17697	−0.07959	−0.31292	−0.51986	−0.70011	−0.85501
8.0	0.61967	1.11565	0.96650	0.71789	0.44061	0.16525	−0.09245	−0.32488	−0.52934	−0.70598	−0.85657
8.5	0.62001	1.12946	0.97080	0.71471	0.43203	0.15361	−0.10489	−0.33624	−0.53819	−0.71134	−0.85784
9.0	0.62023	1.14139	0.97384	0.71082	0.42322	0.14214	−0.11690	−0.34704	−0.54648	−0.71625	−0.85889
9.5	0.62038	1.15173	0.97587	0.70639	0.41429	0.13088	−0.12848	−0.35732	−0.55426	−0.72079	−0.85977
10.0	0.62047	1.16073	0.97706	0.70156	0.40533	0.11987	−0.13963	−0.36709	−0.56159	−0.72499	−0.86051

Table 6 Dimensionless circumferential stresses σ_2 for rotating free annular disks made of aluminum–zirconia FGM

b/a	β										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
4.5	1.94285	1.34254	1.03231	0.75580	0.48346	0.21533	−0.04389	−0.28985	−0.51944	−0.73088	−0.92348
5.0	2.03689	1.35921	1.04436	0.76300	0.48493	0.21205	−0.04985	−0.29593	−0.52313	−0.72991	−0.91597
5.5	2.11207	1.36586	1.04991	0.76496	0.48217	0.20551	−0.05814	−0.30363	−0.52796	−0.72994	−0.90965
6.0	2.17312	1.36655	1.05157	0.76369	0.47677	0.19699	−0.06781	−0.31224	−0.53348	−0.73068	−0.90435
6.5	2.22343	1.36377	1.05089	0.76035	0.46970	0.18727	−0.07824	−0.32133	−0.53938	−0.73193	−0.89990
7.0	2.26542	1.35904	1.04878	0.75568	0.46158	0.17686	−0.08905	−0.33060	−0.54546	−0.73354	−0.89616
7.5	2.30087	1.35332	1.04579	0.75015	0.45280	0.16609	−0.09999	−0.33989	−0.55158	−0.73538	−0.89299
8.0	2.33111	1.34718	1.04226	0.74406	0.44364	0.15516	−0.11090	−0.34907	−0.55765	−0.73737	−0.89029
8.5	2.35715	1.34097	1.03841	0.73760	0.43427	0.14422	−0.12167	−0.35806	−0.56362	−0.73945	−0.88798
9.0	2.37975	1.33488	1.03439	0.73092	0.42482	0.13337	−0.13223	−0.36683	−0.56945	−0.74157	−0.88598
9.5	2.39952	1.32905	1.03026	0.72410	0.41537	0.12267	−0.14254	−0.37534	−0.57511	−0.74371	−0.88424
10.0	2.41693	1.32351	1.02609	0.71723	0.40597	0.11216	−0.15258	−0.38358	−0.58059	−0.74583	−0.88272

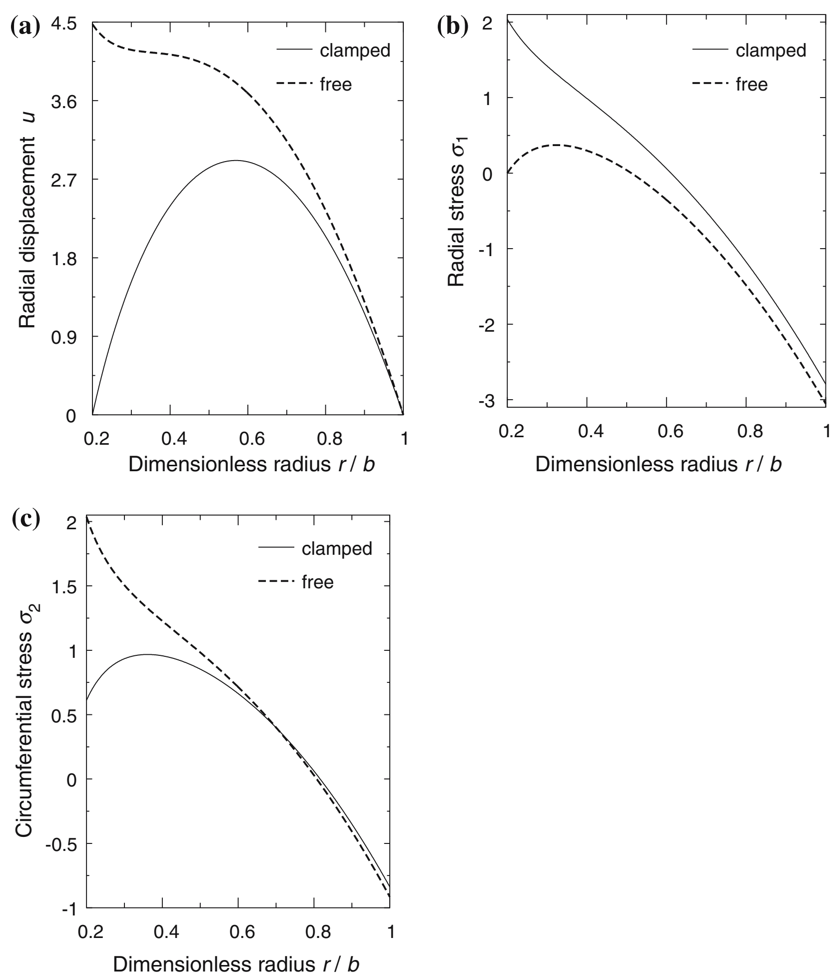

Fig. 2 Dimensionless radial displacement and stresses for the aluminum–zirconia annular disk with rigid casing: (**a**) radial displacement u, (**b**) radial stress σ_1 and (**c**) circumferential stress σ_2

The tabulated results are given for different ratios of the outer-to-inner radii of the FG aluminum–zirconia annular disks. The outer surface of the annular disk is considered to be clamped while the inner surface is either clamped or free. The radial displacement and stresses for FG aluminum–zirconia annular disk are plotted in Fig. 2. For the

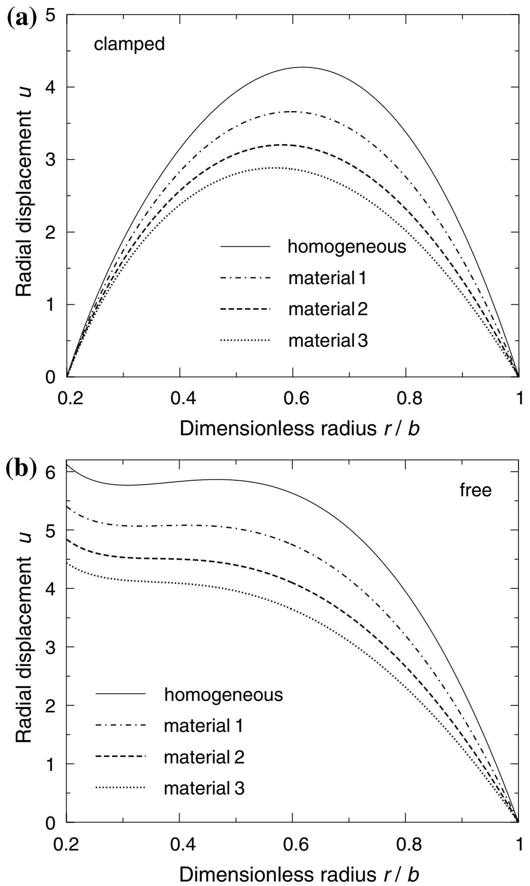

Fig. 3 Dimensionless radial displacement u for the homogeneous and FGM annular disks with rigid casing: (**a**) clamped and (**b**) free

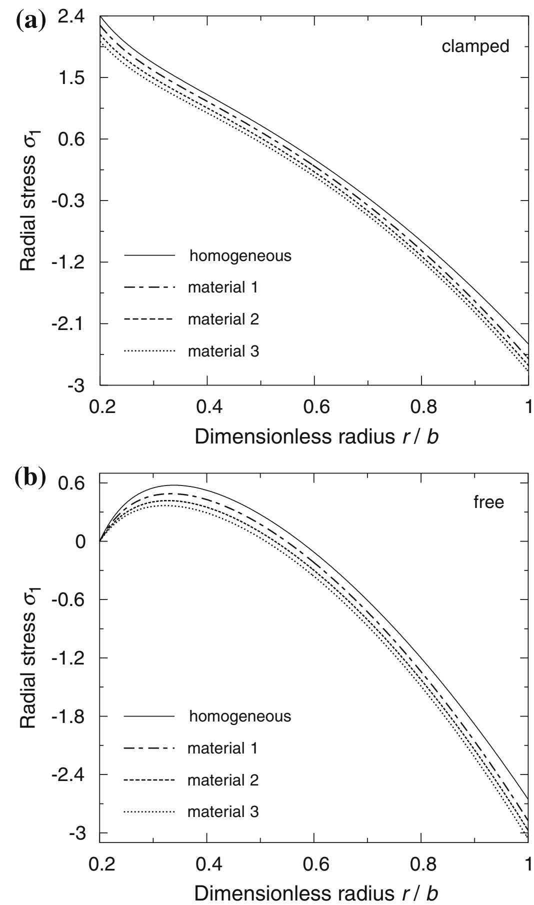

Fig. 4 Dimensionless radial stress σ_1 for the homogeneous and FGM annular disks with rigid casing: (**a**) clamped and (**b**) free

sake of completeness and comparisons, results for homogeneous and different FG materials are presented in Figs. 3–5. It is assumed in all figures that the ratio of outer-to-inner radii is given by $b = 5a$. Note that, for different FG materials considered herein and all values of b/a ratio, the convergence condition of Whittaker's functions, $|k\bar{r}| < 1$, is satisfied [17]. However, for a homogeneous isotropic material the exact solution is easily given.

Tables 1 and 2 show that the radial displacements of clamped disks are smaller than those for free disks irrespective of the values of $\bar{r}$ and $\bar{a}$ ratios. The displacements of free disks are monotonic decreasing in $\bar{r}$ with their maximums occurring at the inner surface $\bar{r} = \bar{a}$. However, the displacements for clamped disks have their maximum values around $\beta = 0.3$ and this, of course, depends on the value of b/a ratio for each disk.

Table 3 shows that the radial stresses for clamped disks decrease as the b/a ratio increases for all values of β. However, the case $\beta = 0$ constitutes an exception, in the sense that the radial stresses increase with the increase of b/a ratio. Table 4 reveals that the radial stresses for all free disks are no longer monotonic increasing in $\bar{r}$ and each has a single maximum around $\beta = 0.1$.

Table 5 shows that the circumferential stresses for all clamped disks are no longer monotonic increasing in $\bar{r}$ and each has a single maximum around $\beta = 0.1$. However, they are monotonic decreasing with their maximums occurring at the inner surfaces of all free disks as given in Table 6.

Figure 2 depicts the displacement and stresses for a rotating FG aluminum–zirconia annular disk as functions of the radial direction of the disk when it is subjected to clamped or free boundary conditions at its inner surface. The clamped disk exhibits the smallest radial displacement u and the greatest radial stress σ_1. The circumferential stress σ_2 of a clamped disk is smaller than that of a free disk for $\bar{r} \leq 0.7$. This situation is then reversed for $\bar{r} > 0.7$. The radial stress σ_1 for the free disk is no longer monotonic increasing in $\bar{r}$ with its maximum occurring at $\bar{r} \cong 0.33$. The circumferential stress σ_2 for the clamped disk is also no longer monotonic increasing in $\bar{r}$ with its maximum occurring at $\bar{r} \cong 0.36$. The maximum value of radial displacement u occurs at $\bar{r} \cong 0.57$ for the clamped disk and, of course, at the inner surface of the free disk.

Concerning the properties of homogeneous and FGMs, Figs. 3–5 show that the displacement and stresses increase

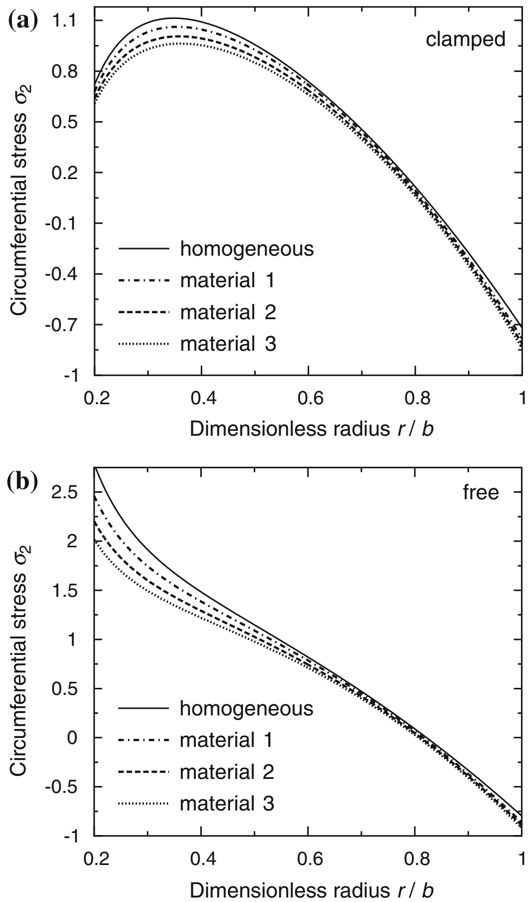

Fig. 5 Dimensionless circumferential stress σ_2 for the homogeneous and FGM annular disks with rigid casing: (**a**) clamped and (**b**) free

with the increase of the material properties ratios. An exception of observation is provided by the homogeneous properties ($E_m = E_c = E$and $\rho_m = \rho_c = \rho$) for which the displacement and stresses are the highest ones. The distributions of the radial displacement for FGMs presented in Fig. 3 along the radial direction of the clamped disk are not parabolic and their maximums occurs at different values of $\bar{r}$. However, the corresponding radial displacement for a homogeneous clamped disk is parabolic with its maximum, of course, at $\bar{r} = 0.5$. Figure 4 shows that the differences between the radial stresses are not changed along the radial direction of the clamped disk and for $\bar{r} > 0.3$ of the free disk. Figure 5 shows that the differences between the circumferential stresses decrease as $\bar{r}$ increases. The maximum differences occur at the inner surface $\bar{r} = \bar{a}$ for the free disk and at $\bar{r} = 0.35$ for the clamped disk.

Conclusions

The problem of rotating functionally graded annular disk is treated and its analytical solution is presented herein. The disk is made from a functionally graded material in which the inner surface of the disk is pure metal material whereas the outer surface of the disk is pure ceramic material. The rotating metal–ceramic disk is analytically studied and closed form solutions for such disks with rigid casing and subjected to clamped or free inner edge conditions are obtained. For the sake of completeness and comparison, results for a homogeneously isotropic disk are presented exactly. The effects due to many parameters on the displacement and stresses are investigated. General results for different rotating FGM annular disks are tabulated for future comparisons.

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